Appendix 1: Discretized Uniaxial PML Equations for 2-D TMz

Using duality on Berenger's 2-D TEz formulation, we arrive at the following finite difference equations in the PML regions where both σ_x and σ_y are nonzero:

$$H_x^{n+1}(i+1/2,j) = e^{-\sigma_y^*(i+1/2,j)\frac{\Delta t}{\mu}} H_x^n(i+1/2,j)$$
$$-\frac{(1-e^{-\sigma_y^*(i+1/2,j)\frac{\Delta t}{\mu}})}{\sigma_y^*(i+1/2,j)\Delta y}\begin{bmatrix} E_{zx}^{n+1/2}(i+1/2,j+1/2)+E_{zy}^{n+1/2}(i+1/2,j+1/2) \\ -E_{zx}^{n+1/2}(i+1/2,j-1/2)-E_{zy}^{n+1/2}(i+1/2,j-1/2)\end{bmatrix}$$

$$H_y^{n+1}(i,j+1/2) = e^{-\sigma_x^*(i,j+1/2)\frac{\Delta t}{\mu}} H_y^n(i,j+1/2)$$
$$-\frac{(1-e^{-\sigma_x^*(i,j+1/2)\frac{\Delta t}{\mu}})}{\sigma_x^*(i,j+1/2)\Delta x}\begin{bmatrix} E_{zx}^{n+1/2}(i-1/2,j+1/2)+E_{zy}^{n+1/2}(i-1/2,j+1/2) \\ -E_{zx}^{n+1/2}(i+1/2,j+1/2)-E_{zy}^{n+1/2}(i+1/2,j+1/2)\end{bmatrix}$$

$$E_{zx}^{n+1/2}(i+1/2,j+1/2) = e^{-\sigma_x(i+1/2,j+1/2)\frac{\Delta t}{\varepsilon}} E_{zx}^{n-1/2}(i+1/2,j+1/2)$$
$$-\frac{(1-e^{-\sigma_x(i+1/2,j+1/2)\frac{\Delta t}{\varepsilon}})}{\sigma_x(i+1/2,j+1/2)\Delta x}\left[H_y^n(i,j+1/2)-H_y^n(i+1,j+1/2)\right]$$

$$E_{zy}^{n+1/2}(i+1/2,j+1/2) = e^{-\sigma_y(i+1/2,j+1/2)\frac{\Delta t}{\varepsilon}} E_{zy}^{n-1/2}(i+1/2,j+1/2)$$
$$-\frac{(1-e^{-\sigma_y(i+1/2,j+1/2)\frac{\Delta t}{\varepsilon}})}{\sigma_y(i+1/2,j+1/2)\Delta y}\left[H_x^n(i+1/2,j+1)-H_x^n(i+1/2,j)\right]$$

It is obvious to see that if $\sigma_x = \frac{\varepsilon}{\mu}\sigma_x^* = 0$, then the equations for E_{zx} and H_y have an undefined fraction (0/0); conversely, if $\sigma_y = \frac{\varepsilon}{\mu}\sigma_y^* = 0$, the equations for E_{zy} and H_x have this same problem. In these regions, we can use standard central differencing in both time and space. If $\sigma_x = \frac{\varepsilon}{\mu}\sigma_x^* = 0$, then the equations for E_{zx} and H_y become:

$$\varepsilon\frac{\partial E_{zx}}{\partial t} = \frac{\partial H_y}{\partial x}$$

$$\mu\frac{\partial H_y}{\partial t} = \frac{\partial E_Z}{\partial x}$$

The resulting finite difference equations are

$$E_{zx}^{n+1/2}(i+1/2, j+1/2) = E_{zx}^{n-1/2}(i+1/2, j+1/2) - \frac{\Delta t}{\varepsilon}\left[H_y^n(i, j+1/2) - H_y^n(i+1, j+1/2)\right]$$

$$H_y^{n+1}(i, j+1/2) = H_y^n(i, j+1/2) - \frac{\Delta t}{\mu}\left[E_z^{n+1/2}(i-1/2, j+1/2) - E_z^{n+1/2}(i+1/2, j+1/2)\right]$$

On the other hand, if $\sigma_y = \frac{\varepsilon}{\mu}\sigma_y^* = 0$, then the equations for E_{zy} and H_x become

$$\varepsilon\frac{\partial E_{zy}}{\partial t} = -\frac{\partial H_x}{\partial y}`$$

$$\mu\frac{\partial H_x}{\partial t} = -\frac{\partial E_Z}{\partial y}$$

The resulting finite difference equations are

$$E_{zy}^{n+1/2}(i+1/2, j+1/2) = E_{zy}^{n-1/2}(i+1/2, j+1/2) - \frac{\Delta t}{\varepsilon}\left[H_x^n(i+1/2, j+1) - H_x^n(i+1/2, j)\right]$$

$$H_x^{n+1}(i+1/2, j) = H_x^n(i+1/2, j) - \frac{\Delta t}{\mu}\left[E_z^{n+1/2}(i+1/2, j+1/2) - E_z^{n+1/2}(i+1/2, j-1/2)\right]$$

Appendix 2: Discretized Uniaxial PML Equations for 2-D TMz

With the given stretched-coordinate metrics S_x, S_y, and S_z in a general uniaxial anisotropic medium, Maxwell's equations can be written as

$$\nabla \times \overline{E} = -j\omega\mu_0 \overline{\overline{S}}\,\overline{H} \tag{1a}$$

$$\nabla \times \overline{H} = -j\omega\varepsilon_0 \overline{\overline{S}}\,\overline{E} \tag{1b}$$

where the matrix $\overline{\overline{S}}$ is given by

$$\overline{\overline{S}} = \begin{bmatrix} \frac{S_y S_z}{S_x} & 0 & 0 \\ 0 & \frac{S_x S_z}{S_y} & 0 \\ 0 & 0 & \frac{S_x S_y}{S_z} \end{bmatrix} \tag{2}$$

With the 2-D TMz, $E_x = E_y = H_z = 0$ and $S_z = 1$.

In region I, $S_x = 1$, $S_y = 1 + \frac{\sigma_y}{j\omega\varepsilon_0}$;

In region II, $S_x = 1 + \frac{\sigma_x}{j\omega\varepsilon_0}$, $S_y = 1$;

In region III, $S_x = 1 + \frac{\sigma_x}{j\omega\varepsilon_0}$, $S_y = 1 + \frac{\sigma_y}{j\omega\varepsilon_0}$.

Region I

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon_0\varepsilon_r (1 + \frac{\sigma_y}{j\omega\varepsilon_0}) E_z \tag{3a}$$

$$-\frac{\partial E_z}{\partial y} = j\omega\mu_0\mu_r (1 + \frac{\sigma_y}{j\omega\varepsilon_0}) H_x \tag{3b}$$

$$\frac{\partial E_z}{\partial x} = j\omega\mu_0\mu_r(\frac{1}{1+\frac{\sigma_y}{j\omega\varepsilon_0}})H_y \tag{3c}$$

Eqn. (3a) and (3b) can be directly discretized with Fourier transform $j\omega = \partial / \partial t$

$$E_z|_{i,j}^{n+1} = \frac{(2\varepsilon_0 - \sigma_y\Delta t)}{(2\varepsilon_0 + \sigma_y\Delta t)}E_z|_{i,j}^{n} + \frac{2\Delta t}{\varepsilon_r(2\varepsilon_0 + \sigma_y\Delta t)}(\frac{H_y|_{i+1/2,j}^{n+1/2} - H_y|_{i-1/2,j}^{n+1/2}}{\Delta x} - \frac{H_x|_{i,j+1/2}^{n+1/2} - H_x|_{i,j-1/2}^{n+1/2}}{\Delta y}) \tag{4a}$$

$$H_x|_{i,j+1/2}^{n+1/2} = \frac{(2\varepsilon_0 - \sigma_y\Delta t)}{(2\varepsilon_0 + \sigma_y\Delta t)}H_x|_{i,j+1/2}^{n-1/2} - \frac{2\varepsilon_0\Delta t}{\mu_0\mu_r(2\varepsilon_0 + \sigma_y\Delta t)}(\frac{E_z|_{i,j+1}^{n} - E_z|_{i,j}^{n}}{\Delta y}) \tag{4b}$$

However, Eqn. (3c) is not linear with frequency. So we apply the two-step method [3] to find H_y.

First let $B_y = \frac{\mu_0\mu_r}{1+\frac{\sigma_y}{j\omega\varepsilon_0}}H_y$, we have

$$B_y|_{i+1/2,j}^{n+1/2} = B_y|_{i+1/2,j}^{n-1/2} + \Delta t(\frac{E_z|_{i+1,j}^{n} - E_z|_{i,j}^{n}}{\Delta x}) \tag{4c-1}$$

Further we use B_y to find H_y,

$$B_y = \frac{\mu_0\mu_r}{1+\frac{\sigma_y}{j\omega\varepsilon_0}}H_y \Rightarrow (1+\frac{\sigma_y}{j\omega\varepsilon_0})B_y = \mu_0\mu_r H_y \Rightarrow j\omega B_y + \frac{\sigma_y}{\varepsilon_0}B_y = j\omega\mu_0\mu_r H_y$$
$$\Rightarrow H_y|^{n+1/2} = H_y|^{n-1/2} + \frac{1}{\mu_0\mu_r}[B_y|^{n+1/2}(1+\frac{\sigma_y\Delta t}{2\varepsilon_0}) - B_y|^{n-1/2}(1-\frac{\sigma_y\Delta t}{2\varepsilon_0})] \tag{4c-2}$$

Region II

Maxwell's equations in region II can be written as

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon_0\varepsilon_r(1+\frac{\sigma_x}{j\omega\varepsilon_0})E_z \tag{5a}$$

$$-\frac{\partial E_z}{\partial y} = j\omega\mu_0\mu_r(\frac{1}{1+\frac{\sigma_x}{j\omega\varepsilon_0}})H_x \tag{5b}$$

$$\frac{\partial E_z}{\partial x} = j\omega\mu_0\mu_r(1+\frac{\sigma_x}{j\omega\varepsilon_0})H_y \tag{5c}$$

Eqn. (5a) and (5c) can be directly discretized with Fourier transform $j\omega = \partial/\partial t$

$$E_z|_{i,j}^{n+1} = \frac{(2\varepsilon_0 - \sigma_x\Delta t)}{(2\varepsilon_0 + \sigma_x\Delta t)}E_z|_{i,j}^{n} + \frac{2\Delta t}{\varepsilon_r(2\varepsilon_0 + \sigma_x\Delta t)}\left(\frac{H_y|_{i+1/2,j}^{n+1/2} - H_y|_{i-1/2,j}^{n+1/2}}{\Delta x} - \frac{H_x|_{i,j+1/2}^{n+1/2} - H_x|_{i,j-1/2}^{n+1/2}}{\Delta y}\right) \tag{6a}$$

$$H_y|_{i+1/2,j}^{n+1/2} = \frac{(2\varepsilon_0 - \sigma_x\Delta t)}{(2\varepsilon_0 + \sigma_x\Delta t)}H_y|_{i+1/2,j}^{n-1/2} + \frac{2\varepsilon_0\Delta t}{\mu_0\mu_r(2\varepsilon_0 + \sigma_x\Delta t)}\left(\frac{E_z|_{i+1,j}^{n} - E_z|_{i,j}^{n}}{\Delta x}\right) \tag{6c}$$

Similarly, we first let $B_x = \frac{\mu_0\mu_r}{1+\frac{\sigma_x}{j\omega\varepsilon_0}}H_x$ and then use two-step method to find H_x,

$$B_x|_{i,j+1/2}^{n+1/2} = B_x|_{i,j+1/2}^{n-1/2} - \Delta t\left(\frac{E_z|_{i,j+1}^{n} - E_z|_{i,j}^{n}}{\Delta y}\right) \tag{6b-1}$$

$$B_x = \frac{\mu_0\mu_r}{1+\frac{\sigma_x}{j\omega\varepsilon_0}}H_x \Rightarrow (1+\frac{\sigma_x}{j\omega\varepsilon_0})B_x = \mu_0\mu_r H_x \Rightarrow j\omega B_x + \frac{\sigma_x}{\varepsilon_0}B_x = j\omega\mu_0\mu_r H_x$$
$$\Rightarrow H_x|^{n+1/2} = H_x|^{n-1/2} + \frac{1}{\mu_0\mu_r}[B_x|^{n+1/2}(1+\frac{\sigma_x\Delta t}{2\varepsilon_0}) - B_x|^{n-1/2}(1-\frac{\sigma_x\Delta t}{2\varepsilon_0})] \tag{6b-2}$$

Region III

Maxwell's equations in region III can be written as

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon_0\varepsilon_r(1+\frac{\sigma_x}{j\omega\varepsilon_0})(1+\frac{\sigma_y}{j\omega\varepsilon_0})E_z \tag{7a}$$

$$-\frac{\partial E_z}{\partial y} = j\omega\mu_0\mu_r\left(\frac{1+\frac{\sigma_y}{j\omega\varepsilon_0}}{1+\frac{\sigma_x}{j\omega\varepsilon_0}}\right)H_x \tag{7b}$$

$$\frac{\partial E_z}{\partial x} = j\omega\mu_0\mu_r\left(\frac{1+\frac{\sigma_x}{j\omega\varepsilon_0}}{1+\frac{\sigma_y}{j\omega\varepsilon_0}}\right)H_y \qquad (7c)$$

All the above three equations need two-step method to find E_z, B_x, and B_y. The final discretized equations are listed as following.

$$\begin{aligned} &D_z = \left(1+\frac{\sigma_y}{j\omega\varepsilon_0}\right)E_z \Rightarrow \\ &E_z|^{n+1} = \frac{(2\varepsilon_0 - \sigma_y\Delta t)}{(2\varepsilon_0 + \sigma_y\Delta t)}E_z|^n + \frac{2\varepsilon_0}{(2\varepsilon_0 + \sigma_y\Delta t)}(D_z|^{n+1} - D_z|^n) \end{aligned} \qquad (8a\text{-}1)$$

$$\begin{aligned} &D_z|^{n+1}_{i,j} = \frac{(2\varepsilon_0 - \sigma_x\Delta t)}{(2\varepsilon_0 + \sigma_x\Delta t)}D_z|^n_{i,j} + \\ &\frac{2\Delta t}{\varepsilon_r(2\varepsilon_0 + \sigma_x\Delta t)}\left(\frac{H_y|^{n+1/2}_{i+1/2,j} - H_y|^{n+1/2}_{i-1/2,j}}{\Delta x} - \frac{H_x|^{n+1/2}_{i,j+1/2} - H_x|^{n+1/2}_{i,j-1/2}}{\Delta y}\right) \end{aligned} \qquad (8a\text{-}2)$$

$$\begin{aligned} &B_x = \frac{1}{1+\frac{\sigma_x}{j\omega\varepsilon_0}}H_x \Rightarrow \left(1+\frac{\sigma_x}{j\omega\varepsilon_0}\right)B_x = H_x \Rightarrow j\omega B_x + \frac{\sigma_x}{\varepsilon_0}B_x = j\omega H_x \\ &\Rightarrow H_x|^{n+1/2} = H_x|^{n-1/2} + \left[B_x|^{n+1/2}\left(1+\frac{\sigma_x\Delta t}{2\varepsilon_0}\right) - B_x|^{n-1/2}\left(1-\frac{\sigma_x\Delta t}{2\varepsilon_0}\right)\right] \end{aligned} \qquad (8b\text{-}1)$$

$$B_x|^{n+1/2}_{i,j+1/2} = \frac{(2\varepsilon_0 - \sigma_y\Delta t)}{(2\varepsilon_0 + \sigma_y\Delta t)}B_x|^{n-1/2}_{i,j+1/2} - \frac{2\varepsilon_0\Delta t}{\mu_0\mu_r(2\varepsilon_0 + \sigma_y\Delta t)}\left(\frac{E_z|^n_{i,j+1} - E_z|^n_{i,j}}{\Delta y}\right) \qquad (8b\text{-}2)$$

$$\begin{aligned} &B_y = \frac{1}{1+\frac{\sigma_y}{j\omega\varepsilon_0}}H_y \Rightarrow \left(1+\frac{\sigma_y}{j\omega\varepsilon_0}\right)B_y = H_y \Rightarrow j\omega B_y + \frac{\sigma_y}{\varepsilon_0}B_y = j\omega H_y \\ &\Rightarrow H_y|^{n+1/2} = H_y|^{n-1/2} + \left[B_y|^{n+1/2}\left(1+\frac{\sigma_y\Delta t}{2\varepsilon_0}\right) - B_y|^{n-1/2}\left(1-\frac{\sigma_y\Delta t}{2\varepsilon_0}\right)\right] \end{aligned} \qquad (8c\text{-}1)$$

$$B_y|^{n+1/2}_{i+1/2,j} = \frac{(2\varepsilon_0 - \sigma_x\Delta t)}{(2\varepsilon_0 + \sigma_x\Delta t)}B_y|^{n-1/2}_{i+1/2,j} + \frac{2\varepsilon_0\Delta t}{\mu_0\mu_r(2\varepsilon_0 + \sigma_x\Delta t)}\left(\frac{E_z|^n_{i+1,j} - E_z|^n_{i,j}}{\Delta x}\right) \qquad (8c\text{-}2)$$

Appendix 3: Discretized CFS PML Equations for 2-D TMz

With the 2-D TMz, $E_x = E_y = H_z = 0$ and S_z = 1.

In region I, $S_x = \kappa_x, S_y = \kappa_y + \dfrac{\sigma_y}{\alpha_y + j\omega\varepsilon_0}$

In region II, $S_x = \kappa_x + \dfrac{\sigma_x}{\alpha_x + j\omega\varepsilon_0}, S_y = \kappa_y$

In region III, $S_x = \kappa_x + \dfrac{\sigma_x}{\alpha_x + j\omega\varepsilon_0},\ S_y = \kappa_y + \dfrac{\sigma_y}{\alpha_y + j\omega\varepsilon_0}$

Start with Maxwell's equations for a 2-D TMz problem, stretched as in [5].

$$1a.(\frac{1}{S_x}\frac{\partial H_y}{\partial x} - \frac{1}{S_y}\frac{\partial H_x}{\partial y}) = \varepsilon_0\varepsilon_r j\omega E_z$$

$$2a. -\frac{\partial E_z}{\partial y}\frac{1}{S_y} = j\omega\mu_0\mu_r H_x$$

$$3a.\frac{\partial E_z}{\partial x}\frac{1}{(S_x)} = j\omega\mu_0\mu_r H_y$$

The inverse of the stretching coefficients are denoted as $\overline{S_x}, \overline{S_y}$. Plugging into the above,

$$1b.(\overline{S_x}\frac{\partial H_y}{\partial x} - \overline{S_y}\frac{\partial H_x}{\partial y}) = \varepsilon_0\varepsilon_r j\omega E_z$$

$$2b. -\frac{\partial E_z}{\partial y}\overline{S_y} = j\omega\mu_0\mu_r H_x$$

$$3b.\frac{\partial E_z}{\partial x}\overline{S_x} = j\omega\mu_0\mu_r H_y$$

In the time domain, the inverse of the stretching coefficient is

$$\overline{S_i(t)} = \frac{\partial(t)}{(\kappa_i)} - \frac{\sigma_i}{\varepsilon_0(\kappa_i)^2}\exp(-(\frac{\sigma_i}{\kappa_i} + \alpha_i)\frac{t}{\varepsilon_0})$$

$$1c.\ \varepsilon_0 \varepsilon_r j\omega E_z = \left(\frac{1}{\kappa_x}\frac{\partial H_y}{\partial x} - \frac{1}{\kappa_y}\frac{\partial H_x}{\partial y} + f(\sigma_x, \overset{x}{H}_y) - f(\sigma_y, \overset{y}{H}_x)\right)$$

$$2c.\ -\frac{\partial E_z}{\partial y}\frac{1}{\kappa_y} - f(\sigma_y, \overset{y}{E}_z) = j\omega\mu_0\mu_r H_x$$

$$3c.\ \frac{\partial E_z}{\partial x}\frac{1}{\kappa_x} + f(\sigma_x, \overset{x}{E}_z) = j\omega\mu_0\mu_r H_y$$

Where $f(\ddot{\sigma}_i, F_j^i) = \frac{\ddot{\sigma}_i}{\kappa_i^2}\int_0^t e^{-(\ddot{\sigma}_i/\kappa_i + \ddot{\alpha}_i)t} F_j^i |^{t-\tau} \, d\tau$ where F_j^i is the spatial derivative WRT *i* of F_j

Discretizing the equations leads to

$$1d.\ \varepsilon_0\varepsilon_r \frac{(E_z|_{i,j}^{n+1} - E_z|_{i,j}^{n})}{\Delta t} = \left(\frac{1}{\kappa_x}\frac{(H_y|_{i+1,j}^{n+1/2} - H_y|_{i,j}^{n-1/2})}{\Delta x} - \frac{1}{\kappa_y}\frac{(H_x|_{i,j+1}^{n+1/2} - H_x|_{i,j}^{n-1/2})}{\Delta y} + \psi_{ezx}^{n+1/2} - \psi_{ezy}^{n+1/2}\right)$$

$$2d.\ -\frac{(E_z|_{i,j+1}^{n} - E_z|_{i,j}^{n})}{\Delta y}\frac{1}{\kappa_y} - \psi_{hxy}^{n+1/2} = \mu_0\mu_r \frac{(H_x|_{i,j}^{n+1/2} - H_x|_{i,j}^{n-1/2})}{\Delta t}$$

$$3d.\ \frac{(E_z|_{i+1,j}^{n} - E_z|_{i,j}^{n})}{\Delta x}\frac{1}{\kappa_x} + \psi_{hyx}^{n+1/2} = \mu_0\mu_r \frac{(H_y|_{i,j}^{n+1/2} - H_y|_{i,j}^{n-1/2})}{\Delta t}$$

$$1e.\ E_z|_{i,j}^{n+1} = E_z|_{i,j}^{n} + \frac{\Delta t}{\varepsilon_0\varepsilon_r}\left(\frac{1}{\kappa_x}\frac{(H_y|_{i+1,j}^{n+1/2} - H_y|_{i,j}^{n-1/2})}{\Delta x} - \frac{1}{\kappa_y}\frac{(H_x|_{i,j+1}^{n+1/2} - H_x|_{i,j}^{n-1/2})}{\Delta y} + \psi_{ezx}^{n+1/2} - \psi_{ezy}^{n+1/2}\right)$$

ψ

$$2e.\ \left(-\frac{(E_z|_{i,j+1}^{n} - E_z|_{i,j}^{n})}{\Delta y}\frac{1}{\kappa_y} - \psi_{hxy}^{n+1/2}\right)\frac{\Delta t}{\mu_0\mu_r} + H_x|_{i,j}^{n-1/2} = H_x|_{i,j}^{n+1/2}$$

$$3e.\ \left(\frac{(E_z|_{i+1,j}^{n} - E_z|_{i,j}^{n})}{\Delta x}\frac{1}{\kappa_x} + \psi_{hyx}^{n+1/2}\right)\frac{\Delta t}{\mu_0\mu_r} + H_y|_{i,j}^{n-1/2} = H_y|_{i,j}^{n+1/2}$$

Where the auxiliary update variables ψ are defined as below.

$$\psi_{ezy}^{n+1/2} = e^{-(\sigma_x/\kappa_x + \alpha_x)\frac{\Delta t}{\varepsilon_0}}\psi_{ezy}^{n+1/2} + a_x \frac{(H_y|_{i+1,j}^{n+1/2} - H_y|_{i,j}^{n+1/2})}{\Delta x}$$

$$\psi_{ezx}^{n+1/2} = e^{-(\sigma_y/\kappa_y + \alpha_y)\frac{\Delta t}{\varepsilon_0}}\psi_{ezx}^{n+1/2} + a_y \frac{(H_x|_{i,j+1}^{n+1/2} - H_x|_{i,j}^{n+1/2})}{\Delta y}$$

$$\psi_{hxz}^{n+1/2} = e^{-(\sigma_y/\kappa_y + \alpha_y)\frac{\Delta t}{\varepsilon_0}} \psi_{hxy}^{n+1/2} + a_y \frac{(E_z |_{i,j+1}^{n+1/2} - E_z |_{i,j}^{n+1/2})}{\Delta y}$$

$$\psi_{hyz}^{n+1/2} = e^{-(\sigma_x/\kappa_x + \alpha_x)\frac{\Delta t}{\varepsilon_0}} \psi_{hyx}^{n+1/2} + a_x \frac{(E_z |_{i+1,j}^{n+1/2} - E_z |_{i,j}^{n-1/2})}{\Delta x}$$

Where $a_i = \dfrac{\sigma_i}{\sigma_i \kappa_i + \kappa_i^2 \alpha_i}(e^{-(\sigma_i/\kappa_i + \alpha_i)\frac{\Delta t}{\varepsilon_0}} - 1)$

There are three different PML regions, thus, three different combinations for values of $\overline{S_x}, \overline{S_y}$

Region 1: $S_x = \kappa_x -> f(\sigma_x, N) = 0 -> \psi_{ezx} = \psi_{hyx} = 0$.

Region 2: $S_y = \kappa_y -> f(\sigma_y, N) = 0 -> \psi_{ezy} = \psi_{hxy} = 0$.

So the Region III derivation is shown above, the final descretized equations for region 1 and 2 are below.

Region I:

$$E_z |_{i,j}^{n+1} = E_z |_{i,j}^{n} + \frac{\Delta t}{\varepsilon_0 \varepsilon_r}\left(\frac{1}{\kappa_x}\frac{(H_y |_{i+1,j}^{n+1/2} - H_y |_{i,j}^{n-1/2})}{\Delta x} - \frac{1}{\kappa_y}\frac{(H_x |_{i,j+1}^{n+1/2} - H_x |_{i,j}^{n-1/2})}{\Delta y} - \psi_{ezy}^{n+1/2}\right)$$

$$\left(-\frac{(E_z |_{i,j+1}^{n} - E_z |_{i,j}^{n})}{\Delta y}\frac{1}{\kappa_y} - \psi_{hxy}^{n+1/2}\right)\frac{\Delta t}{\mu_0 \mu_r} + H_x |_{i,j}^{n-1/2} = H_x |_{i,j}^{n+1/2}$$

$$\left(\frac{(E_z |_{i+1,j}^{n} - E_z |_{i,j}^{n})}{\Delta x}\frac{1}{\kappa_x}\right)\frac{\Delta t}{\mu_0 \mu_r} + H_y |_{i,j}^{n-1/2} = H_y |_{i,j}^{n+1/2}$$

Region II:

$$E_z |_{i,j}^{n+1} = E_z |_{i,j}^{n} + \frac{\Delta t}{\varepsilon_0 \varepsilon_r}\left(\frac{1}{\kappa_x}\frac{(H_y |_{i+1,j}^{n+1/2} - H_y |_{i,j}^{n-1/2})}{\Delta x} - \frac{1}{\kappa_y}\frac{(H_x |_{i,j+1}^{n+1/2} - H_x |_{i,j}^{n-1/2})}{\Delta y} + \psi_{ezx}^{n+1/2}\right)$$

$$\left(-\frac{(E_z |_{i,j+1}^{n} - E_z |_{i,j}^{n})}{\Delta y}\frac{1}{\kappa_y}\right)\frac{\Delta t}{\mu_0 \mu_r} + H_x |_{i,j}^{n-1/2} = H_x |_{i,j}^{n+1/2}$$

$$\left(\frac{(E_z |_{i+1,j}^{n} - E_z |_{i,j}^{n})}{\Delta x}\frac{1}{\kappa_x} + \psi_{hyx}^{n+1/2}\right)\frac{\Delta t}{\mu_0 \mu_r} + H_y |_{i,j}^{n-1/2} = H_y |_{i,j}^{n+1/2}$$

We can note here that the third equation for Region I and the second equation for Region II are very similar to the update equations for Hy and Hx, respectively, in free space. Thus, high reflections are introduced in these regions and PML does not effectively absorb the wave; the field is reflected back as if in free space.